Numerical upscaling and iMSFV Method for Stokes-Brinkman and Stokes-Darcy equations

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Abstract

Numerical upscaling for Stokes-Brinkman and a multiscale method for Stokes-Darcy problem is considered. Both methods allow to compute flow and transport in porous media with geometries that can be too expensive to be solved on a fine scale computational grid. A set of coarse and fine grids are used, where the fine grid resolves the geometry accurately but requires very high computational efforts. Each coarse grid cell is a union of fine grid cells. The first algorithm focuses on a subgrid resolution approach for solving the Stokes-Brinkman system. The approach allows to account for finer scale details by solving auxiliary problems in appropriately chosen “quasi porous” blocks on a coarse grid. The method is based on numerical upscaling and includes a systematic and a careful procedure of modifying and updating the coefficients of the Navier-Stokes-Brinkmann system in the chosen blocks. The second algorithm is an iterative-multiscale-finite-volume(iMSFV) method for Stokes-Darcy system, describing flow in plain and in porous media. iMSFV method is extended for solving the pressure correction equations in a SIMPLE-type algorithmic framework. The coarse scale ensures global coupling and the fine scale ensures appropriate resolution. The basis functions and correction functions correspond to the Stokes equations in the fluid domain. Simultaneously, for the porous domain, Darcy equation is employed.

1. Porous media flow: industrial applications

Filteration is a multiscale phenomenon. A filter consists of a filter housing with inlet(s) and outlet(s) separated by one or multiple filtering media.

Figure 1: Examples of complicated filter geometries. Combi Filter(left) consists of multiple porous media. Pleated filter (right) has a complicated shape of the filter media. Very high resolution is required to resolve such geometrical details.

Accurate resolving of filter geometries and filtering media requires enormously high computational efforts. To overcome this bottleneck, the so-called localization approach is considered and upscaling/multiscale methods merit close examination. Subgrid method represents the large scale system by incorporating finer details in an average sense, whereas MSFV method incorporates them via fine scale fluxes. Independent local problems are solved on the fine scale, solutions of which are fed into the coarse global system. The coarse scale enforces global coupling whereas the fine scale resolves the problem heterogeneities.

Figure 2: Schematic view of a filter(top). Different levels of resolution(bottom) which require extremely high computational resources.

Stokes-Brinkmann system of equations is one of the ways to describe flow through filters for complicated geometries.

The equations hold in domain \( \Omega = \Omega_1 \cup \Omega_p \),

\[
-\nabla \cdot ( \rho \nu \vec{u} ) + \rho \nabla \vec{p} + \vec{f} = 0
\]

Another formulation for describing flow in plain and porous media is the Stokes-Darcy equations, whereby the Darcy equation governs the flow in \( \Omega_p \), and the Stokes equations govern flow in \( \Omega \). The equations are considered in a coupled manner to form a system of partial differential equations

\[
-\nabla \cdot ( \rho \nu \vec{u} ) = -\nabla \vec{p} \quad \text{in} \ \Omega_p
\]

where \( p \) stands for velocity pressure respectively and \( \rho \) denotes the density and viscosity, respectively. \( \lambda \) is the permeability tensor of the porous medium.

2. SIMPLE-type decoupling method

Systems (1) and (2) are solved by decoupling pressure and velocity. Here we consider the decoupling method only for the Stokes equations, without the Brinkmann term/Darcy equations. The operators corresponding to the discretized diffusion term and the gradient term in the momentum equation are denoted by \( D \) and \( G \) respectively. Analogously, \( C \) denotes the discrete divergence operator. The decoupling method for the Stokes equations can be written as

\[
-D\vec{u}^n = G\vec{p}^n
\]

where \( \vec{u} \) denotes the velocity predictions. The first equation is solved with respect to velocities, using the old values of the pressure gradient. Divergence of equation (4) and using the continuity equation results in a poisson type equation for pressure correction, denoted by \( \vec{p}^* \).

\[
C^T\vec{u}^n = -G^T\vec{p}^* + \vec{q}
\]

After the Poisson type equation for pressure correction is solved, pressure is updated by \( p^{n+1} = p^n + q \) and the velocity by \( \vec{u}^{n+1} = \vec{u}^n + \lambda\vec{q} \).

For porous media flow, in the case of Stokes-Brinkman/Stokes-Brinkmann problem, \( \lambda = \lambda_1 \) in \( \Omega_1 \) and \( \lambda = \frac{K}{\mu} = \frac{(k_1 \times k_2 \times k_3)}{\mu} \) in \( \Omega_p \). The numerical solution of such kind of systems with varying coefficients require very high spatial resolution for complex geometries, which lead to enormously high computational resources. The speed with which these matrix equations are solved and the amount of memory required to solve them pose serious bottlenecks in performing realistic large scale simulations. Additionally, accuracy is a fundamental issue, not only in terms of solvability but also accurate modeling that incorporates particularly the small scale heterogeneity.

3. Subgrid resolution approach

Key ingredients to the proposed method are two resolution scales: coarse scale and a refined fine scale. The method starts with the coarse blocks and selects quasi porous auxiliary blocks which look for heterogeneity in the porous medium or for complicated fine scale geometry.

A block can contain one or many coarse grid elements, on which system (1) is solved. A new permeability is computed for each of such blocks using the solution. The usage of correct parameters is critical in solving auxiliary local problems which are then used to compute new/modifed permeability for a quasi porous block. The computed permeabilities are input for the global coarse scale problem. The method emphasizes on the determination of upscaled permeabilities for use in subsequent coarse scale global simulations.

Figure 4: Results of simulations: filter geometry on the coarse grid (left), fine grid (middle) and subgrid (right). Coefficients are updated for the quasi-porous block (orange right) for which an upscaled permeability is computed. Coarse problem is solved with the modified permeability.

4. Multiscale Finite Volume Approach

To illustrate this approach, system (2) is used. The velocity predictions are first obtained for the Stokes region while using the old step velocity predictions for the Darcy regions. An elliptic equation is constructed for the pressure corrections. MSFV method is employed to solve this equation. The approach employs a set of two coarse grids, namely the primal and a dual coarse grids, in addition to the underlying fine grid. Local auxiliary problems for the basis and correction functions are solved separately in each block of the dual coarse grid, where Stokes-Darcy equations are discretized on the underlying fine grid.

Figure 5: Basis and Correction functions computed on a dual cell consisting of fluid and porous regions. The solutions of the auxiliary problems are used within the framework of Multiscale Finite Element Method to build a coarse grid discrete system. Basis and Correction functions are computed for each dual coarse block and the fluxes are approximated onto the block boundaries. Note that the discretization on the fine grid is an important component in solving the auxiliary problems in the dual coarse blocks.

Figure 6: Results of simulations: pressure (left) and velocity (right). Black arrows are velocity vectors.

References
