How to solve difficult cutting and packing problems? By semi-infinite optimization!

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Tasks in cutting and packing optimization

The tasks in ***cutting and packing*** (***C&P***) optimization are the following:

- Arrange *as many* items in a container as possible.
- Arrange all items in a container of *minimal size* or in *as few* containers as possible.
- Cut items from one or more containers while *minimizing waste*.

For specific item and container shapes, there is a huge variety of models and solution methods. But the number of algorithms rapidly decreases as the shape of the items and/or the container becomes more complicated (see Fig. 1 & 2).

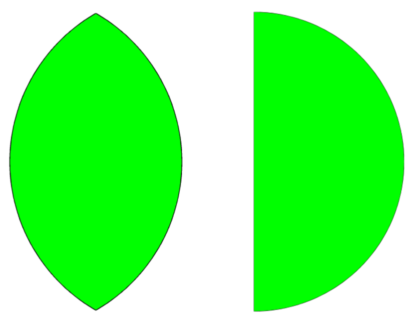


Fig. 1: Two non-standard items

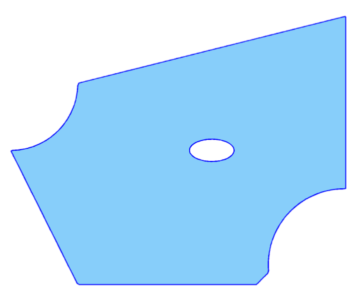


Fig. 2: Convex polygon with surface and internal cavities as container

From set-theoretic conditions to semi-infinite constraints

There are two types of conditions in any C&P problem:

1) each item $B_k(\mathbf{x})$ must entirely lie in the container $C(\mathbf{x})$ (***containment conditions***):

$$\boxed{B_k(\mathbf{x}) \subseteq C(\mathbf{x}) \;\forall k}$$

2) the items do not overlap (***non-overlapping conditions***):

$$\boxed{B_{k_1}(\mathbf{x}) \cap \mathrm{int}(B_{k_2}(\mathbf{x})) = \emptyset \;\forall k_1, k_2 : k_1 < k_2}$$

where some or all objects depend on the decision variables $\mathbf{x}$.

These intractable set-theoretic arrangement conditions can be transferred into ***semi-infinite constraints*** in the following way (see [KMS15] and the references therein for details and further reformulations):

Reformulation of 1):

If $C(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^n \mid c_i(\mathbf{x},\mathbf{y}) \leq 0,\; i \in I\}$,
then $B_k(\mathbf{x}) \subseteq C(\mathbf{x})$
$\Leftrightarrow \quad c_i(\mathbf{x},\mathbf{y}) \leq 0 \;\forall\, \mathbf{y} \in B_k(\mathbf{x}),\;\; i \in I.$

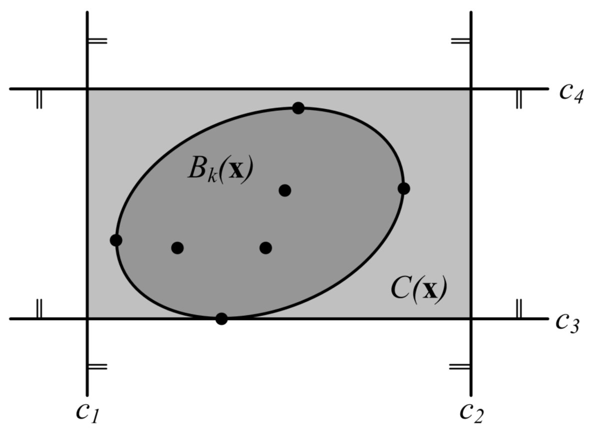


Fig. 3: Containment of an elliptical item in a rectangular container

Reformulation of 2):

$B_{k_1}(\mathbf{x}) \cap \mathrm{int}(B_{k_2}(\mathbf{x})) = \emptyset$
$\Leftrightarrow \quad \exists\, \boldsymbol{\eta} \neq \mathbf{0}$ and β :
$\boldsymbol{\eta}^T\mathbf{y} - \beta \leq 0 \;\forall\, \mathbf{y} \in B_{k_1}(\mathbf{x})$
$\boldsymbol{\eta}^T\mathbf{z} - \beta \geq 0 \;\forall\, \mathbf{z} \in B_{k_2}(\mathbf{x})$

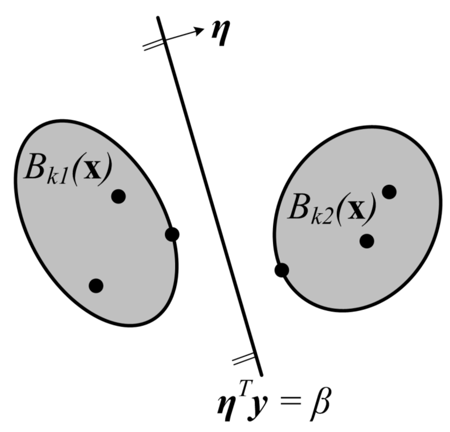


Fig. 4: Two non-overlapping elliptical items

Additional requirements

Important practical requirements can be easily integrated into the model (see [KMS15] and the references therein):

- **Guillotine arrangement** of items for sectioning the container successively by straight-lined, end-to-end cuts, e.g. by a circular saw (see Fig. 5),
- a **minimal distance** between the items for saw kerfs (see Fig. 6), and
- **quality requirements** (avoid certain defects in selected item parts) in order to guarantee a good quality of an item or to improve its quality (see Fig. 7).

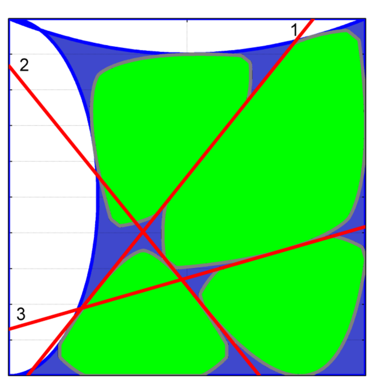


Fig. 5: Guillotine arrangement of four gemstone-like items

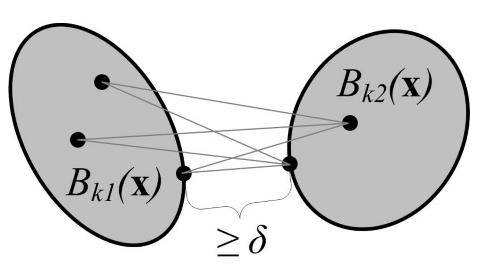


Fig. 6: Minimal distance between two elliptical items

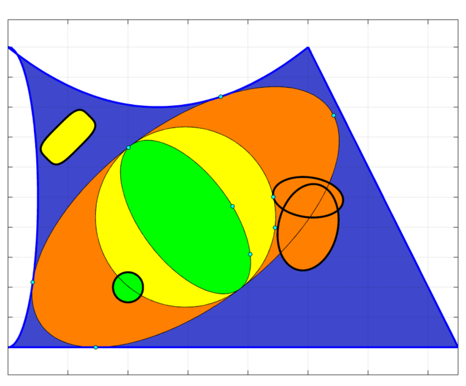


Fig. 7: Quality requirements for an elliptical item

Applied to gemstone cutting

In gemstone cutting an irregularly shaped raw gem having surface flaws and being interspersed with inclusions must be cut into blanks such that the total value of the manufacturable precious gems is maximized (see Fig. 8). We considered three of the four price-setting criteria, the so-called the ***4 C's***: *carat/volume*, *clarity*, *cut*, and color [MM12].

The main challenges are:

- *irregular* raw gem and defects,
- *wide range* of different shapes and cuts,
- *size-dependent* faceting, and
- taking *aesthetic sensibility* into account.

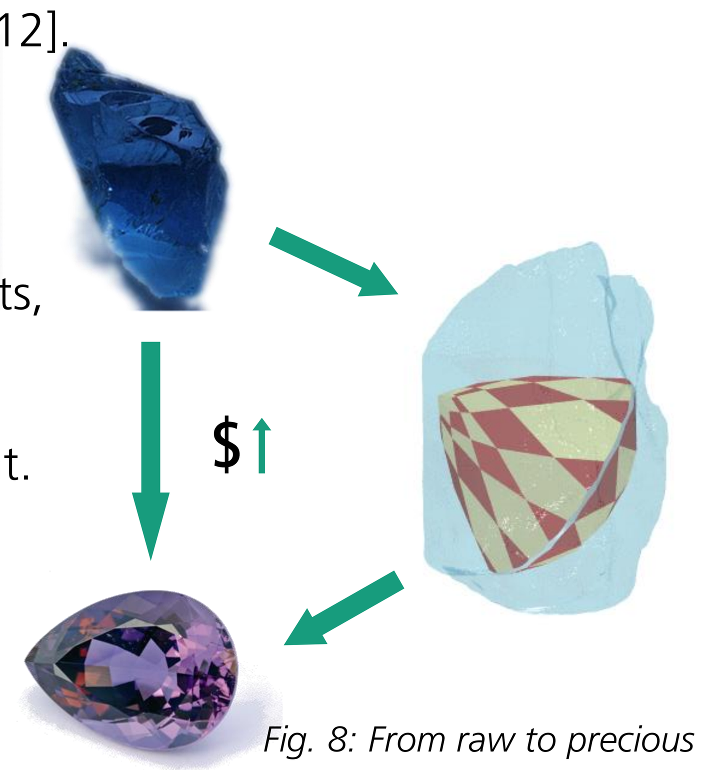
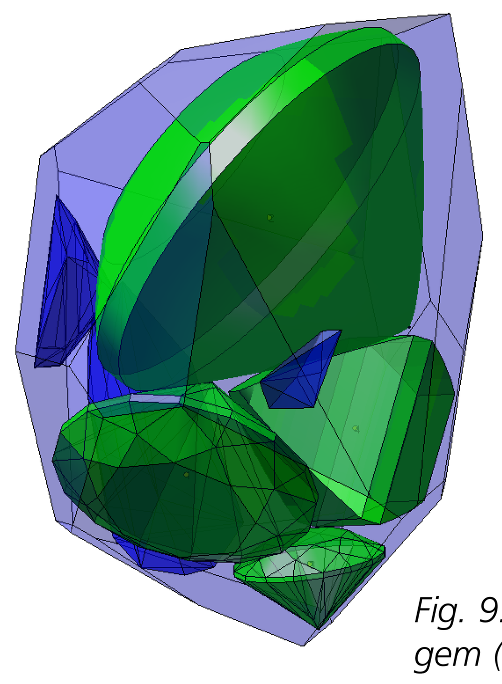


Fig. 8: From raw to precious gem



By improving as well as developing new semi-infinite optimization techniques we demonstrated, that *realistic* problem instances can be solved on a *standard* PC in *reasonable* time (see Fig. 9 & [KMS15] for details).

Fig. 9: Two Brilliants, an Oval, and a Baguette (all green) in an irregular raw gem (blue) maximizing total volume and avoiding defects (darkblue)

The work resulted not only in many publications, but also in a ***software product*** (see Fig. 10) and in the ***first fully automated gemstone production process*** (see Fig.11) [ITWM, FP09].

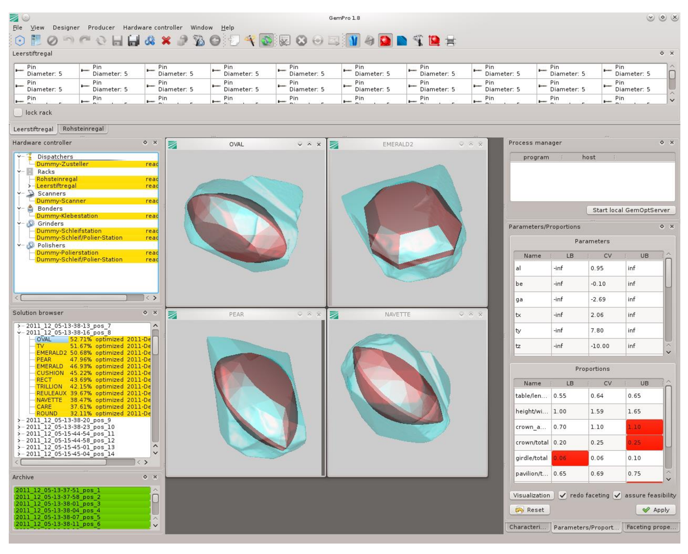


Fig. 10: GUI of GemOpt software

Fig. 11: Machine prototype

Summary and outlook

On the one hand, semi-infinite optimization allows modelling and solving of difficult C&P problems in the first place. And on the other hand it provides a ***unified approach*** to solve C&P problems in general. Furthermore, by this way one can also solve other geometric optimization problems like layout or positioning problems.

Future topics of research are:

- freeform (water-/laserjet) cutting (see Fig. 12),
- computation of globally optimal arrangements, and
- automation of raw gem sectioning.

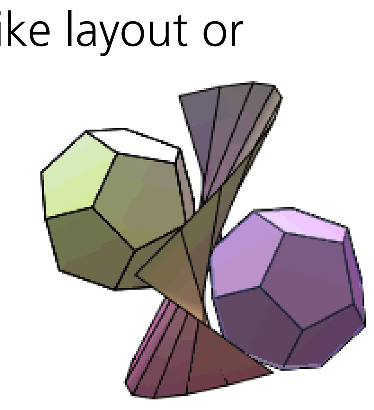


Fig. 12: Freeform jet cutting

References (selected)

[ITWM] **Project website**: http://www.itwm.fraunhofer.de/abteilungen/optimierung/optimierung-im-virtual-engineering/optimale-verwertung-von-farbedelsteinen.html

[KMS15] **Book chapter**: K.-H. Küfer, V. Maag, and J. Schwientek (2015): *Maximale Materialausbeute bei der Edelsteinverwertung*. Chapter 8 in H. Neunzert and D. Prätzel-Wolters (eds.): *Mathematik im Fraunhofer-Institut*. pp. 239-301 (in German, End of 2015 in English)

[MM12] **Podcast**: Being on the Cutting Edge. Mathematical Moments, AMS, http://www.ams.org/samplings/mathmoments/mm94-diamond-podcast

[FP09] **Press release** & **video clip**: www.itwm.fraunhofer.de/en/gemstones

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